Spontaneous magnetization of the Ising model on the Sierpinski carpet fractal, a rigorous result

2003 J. Phys. A: Math. Gen. 36 1593

(http://iopscience.iop.org/0305-4470/36/6/305)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **36** (2003) 1593–1604 PII: S0305-4470(03)56197-2

Spontaneous magnetization of the Ising model on the Sierpinski carpet fractal, a rigorous result

A Vezzani

Dipartimento di Fisica, Istituto Nazionale Fisica della Materia (INFM), Università di Parma, parco Area delle Scienze 7A 43100, Parma, Italy

E-mail: vezzani@fis.unipr.it

Received 12 November 2002
Published 29 January 2003
Online at stacks.iop.org/JPhysA/36/1593

Abstract

We give a rigorous proof of the existence of spontaneous magnetization at finite temperature for the Ising spin model defined on the Sierpinski carpet fractal. The theorem is inspired by the classical Peierls argument for the two-dimensional lattice. Therefore, this exact result proves the existence of spontaneous magnetization for the Ising model in low-dimensional structures, i.e. structures with dimension smaller than 2.

PACS numbers: 75.10.Hk, 61.43.Hv, 05.70.Fh, 05.45.Df

1. Introduction

The study of the physical models on the Sierpinski carpet [1] has been the focus of several works in the last two decades. However, most of these papers are based on approximate renormalization group techniques [2, 3], on perturbative expansions [4] or on numerical simulations [5, 6]. Thus, very few rigorous results are known for this self-similar structure. Indeed, the Sierpinski carpet is an infinitely ramified fractal, where it is not possible to apply the exact decimation techniques, which allow us to solve many models, defined on finitely ramified fractals, such as the Sierpinski gasket [7, 8].

A relevant theorem is given by an upper and a lower bound on the return probability of a simple random walk [9]. These bounds prove that the spectral dimension [10] of the Sierpinski carpet is smaller than 2. Hence, from the generalized Mermin–Wagner theorem [11], the continuous symmetry spin models do not present spontaneous magnetization. On the other hand, there are no general criteria relating the behaviour of the discrete symmetry models, in particular Ising, with a simple parameter, describing the topology of the space where the model is defined. The main known results consider the Euclidean lattices, where spontaneous magnetization is present if the dimension is $\geqslant 2$, and the exactly decimable fractals, which never present spontaneous magnetization [7].

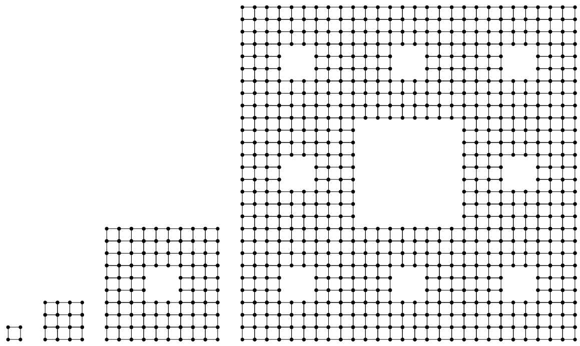

Figure 1. The first four generations of the Sierpinski carpet graph. The sites of the graph (denoted by points) represent the spins and the links (denoted by lines) represent the interactions.

However, the approximate calculations [2, 4] and the numerical simulations [5] suggest that an Ising spin system on the Sierpinski carpet is spontaneously magnetized at finite temperature. Recently, this property has been proved in [12], where it is shown that it follows from the behaviour of percolation on the Sierpinski carpet. Here we give an alternative proof inspired by the classical Peierls argument [13] for the Ising model on the two-dimensional lattice and, in particular, by the version given in [14]. These theorems are the first exact results showing the existence of an Ising transition in a low-dimensional structure, i.e. in a structure with spectral dimension smaller than 2.

We will consider the Sierpinski carpet graph illustrated in figure 1, even if the proof can be generalized to different infinitely ramified carpets. With each site of the graph we associate an Ising spin and with each link a ferromagnetic interaction. We call p_i^- the probability for the magnetization of the spin i to be smaller than 1 and we show that, at low enough temperature, $p_i^- < \epsilon < 1/2$ for each site i. The theorem is based on a low-temperature loop expansion of this probability and on an estimate of the number of self-avoiding on the sites (SOS) loops.

The main difference from the two-dimensional case is that now the space, called a dual graph, where the loops are defined, has unbounded connectivity, namely, the coordination (number of nearest neighbours) of a site is finite but not bounded. Hence, the proof requires more complicated mathematical techniques. Note that the standard Peierls argument results are rather simple, as the dual of the two-dimensional lattice is again a two-dimensional lattice. In spite of the unbounded connectivity, here we will be able to estimate the number of SOS loops because of an important property of the dual graph, which will be called *sparse high connectivity*, i.e. sites with large coordination are far apart. In particular, when the coordination is greater than 4×3^m, the sites are at a distance of at least 2^{m+1} in the intrinsic metric of the dual graph.

Peierls argument is valuable because of the insight it gives into why the Ising model is magnetized. Hence, we believe that this proof could be useful to find a general criterion for the existence of spontaneous magnetization on a generic discrete structure. Note that Peierls argument has already been used to show the existence of an Ising transition in inhomogeneous systems, in particular in the case of diluted random graphs above the percolation thresholds [15]. We start by introducing some definitions and notation.

2. Definitions and notation

Let us consider the Nth generation of a Sierpinski carpet. On each site i we define an Ising spin $s_i = \pm 1$ and we fix the spins on the boundary to the value +1 (see figure 2). The energy

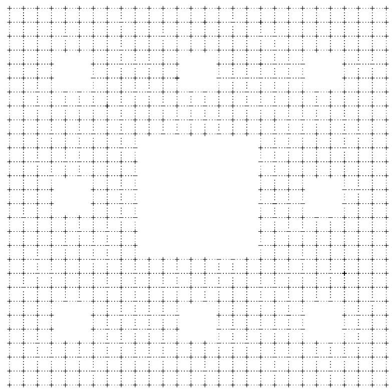

Figure 2. A spin configuration of the Ising spin model defined on the fourth generation of the Sierpinski carpet. The spins of the boundary are fixed to be +1. The dotted lines represent the interactions.

of the spin system is given by

$$-\frac{1}{2}\sum_{\langle ij\rangle} s_i s_j \tag{1}$$

where the sum runs over all the pairs $\langle ij\rangle$ of nearest neighbour sites, i.e. sites connected by a link. In the canonical ensemble, the probability for the magnetization of the spin i to be smaller than 1 is

$$p_i^- = \frac{\sum_{\{s_1,\ldots,s_N\}}^{s_i=-1} \mathrm{e}^{\beta/2\sum_{\langle ij\rangle} s_i s_j}}{\sum_{\{s_1,\ldots,s_N\}} \mathrm{e}^{\beta/2\sum_{\langle ij\rangle} s_i s_j}}. \tag{2}$$

In the denominator we sum over all possible spin configurations with +1 boundary conditions. In the numerator we sum over all configurations with +1 boundary conditions and the spin i fixed to -1. We will prove the spontaneous magnetization of the system by showing the existence of a temperature $\bar{\beta}$ such that, in the thermodynamic limit, $\forall\beta > \bar{\beta}$, $p_i^- < \epsilon < 1/2$.

The theorem will be proved in the following steps. In section 3 we introduce the dual space and the loop expansion of equation (2). With this expansion, we prove that p_i^- can be overestimated by the sum of $\exp(-\beta|l|)$ over all SOS loops enclosing i ($|l|$ is the length of the loop l).

In section 4 we introduce a tree representation of the SOS loops of length $|l|$ enclosing i and passing the site α. This representation allows us to overestimate the number of loops as $\tilde{Z}_{\bar{l}_i^\alpha}\prod_{\gamma\in\bar{l}'^\alpha_i}\tilde{z}_\gamma$. $\bar{l}_i^\alpha$ is a SOS loop. The product runs over the sites belonging to $\bar{l}_i^\alpha$ excluding the site of highest coordination. $\tilde{Z}_{\bar{l}_i^\alpha}$ is the coordination number of the site representing, in the tree, the site of highest coordination in $\bar{l}_i^\alpha$.

In section 5 we define the sparse high connectivity property and, using this property, we show that $\prod_{\gamma\in\bar{l}'^\alpha_i}\tilde{z}_\gamma \leqslant \exp(K|l|)$ where K is a suitable real constant.

In section 6 we prove that $\tilde{Z}_{\bar{l}_i^\alpha} \leqslant 10|l|^{\log_2(3)}$. In particular, we exploit the fact that $\tilde{Z}_{\bar{l}_i^\alpha}$ is smaller than the number of nearest neighbours, of the site of highest coordination, which can be crossed by a loop of length $|l|$ and surrounding i.

In section 7 we show that the number of sites which can be crossed by a SOS loop of length $|l|$ and surrounding i is smaller than $\pi(8|l|)^{2\log_2(3)}$. This last estimate is proved by

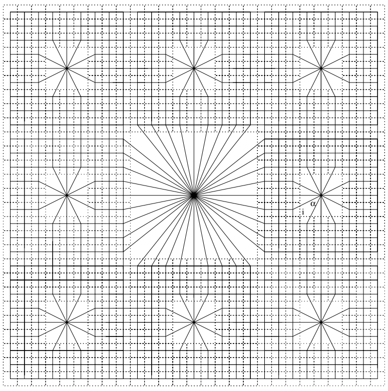

Figure 3. The dashed line denotes the Sierpinski carpet and the continuous line the dual graph. The site i belongs to the carpet, and α to the dual graph.

evaluating the maximum Euclidean distance that can be covered in the dual lattice by a loop of $|l|$ steps. Finally, we will obtain

$$p_i^- \leqslant \sum_{|l|=0}^{\infty} C|l|^{3\log_2(3)}\,\mathrm{e}^{(K-\beta)|l|} \tag{3}$$

where C is a suitable positive constant. From equation (3) we have that $p_i^- < \epsilon < 1/2$ for small enough temperature.

Besides the sparse high coordination property, in the proof we strongly use the fact that the dual lattice is embedded in a two-dimensional Euclidean space and the fact that the plaquettes of the carpet are convex sets of this space.

3. Loop expansion

In analogy with Peierls argument, we write the Hamiltonian of the system in terms of a loop expansion. These loops belong to the dual graph of the discrete structure where the model is defined. In particular, the dual graph of the Sierpinski carpet is obtained by introducing a site α for each plaquette of the carpet. Then, two sites are connected if and only if there exists a link of the carpet separating the plaquettes relevant to the sites we are considering. Hence, there is a one-to-one correspondence between the links of the carpet and the links of the dual. We will call V_d the set of the sites of the dual graph and E_d the set of the links. Note that the dual graph has unbounded connectivity; hence, it must be studied with a certain caution. The length (number of links) of the shortest walk connecting two sites $\in V_d$ will be called the chemical distance between them. In figure 3 we illustrate the Sierpinski carpet and its dual graph.

Let us introduce a one-to-one correspondence between the spin configurations and the self-avoiding loops of the dual graph (self-avoiding on the links). A spin configuration with +1 boundary conditions is identified by the links of the carpet which separate the spins with +1 magnetization and the spins with -1 magnetization. Since, by definition, each link of the dual graph corresponds exactly to a link of the carpet, we obtain a relation between the spin configuration and a subset $L \subseteq E_d$. It is easy to show that the number of links in L adjacent to the same site α is even. Therefore, L can always be obtained as a superposition of self-avoiding (on the links) loops. If in L there are more than two links adjacent to the same site, the decomposition of L into loops is not unique.

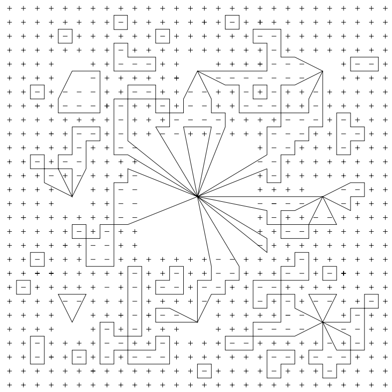

Figure 4. A spin configuration on the Sierpinski carpet and the corresponding link set $L \subset E_d$.

On the other hand, let us call L any subset $\subseteq E_d$ such that for any $\alpha \in V_d$ the number of links in L adjacent to α is even, i.e. L can be decomposed into loops (we will call $\bar{E}_L$ the set of all possible L). In a unique way we can associate with L a spin configuration with +1 boundary conditions, by fixing to +1 the spins enclosed by an even number of loops and to -1 the spins enclosed by an odd number of loops. In figure 4 we illustrate a spin configuration and its corresponding $L \subseteq E_d$.

The energy of the spin configuration can be written in terms of the corresponding set of links L. Indeed

$$H = -\frac{1}{2}\sum_{\langle ij\rangle} s_i s_j = -|E_d| + \frac{1}{4}\sum_{\langle ij\rangle}(s_i - s_j)^2 \tag{4}$$

where $|E_d|$ is the total number of links of the dual graph. A link gives a contribution 1 to the Hamiltonian (4), if it connects opposite spins, and gives zero, if it connects parallel spins. Dropping the irrelevant additive constant $|E_d|$, we obtain $H = |L|$, where $|L|$ is the number of links in the set $L \in \bar{E}_L$ corresponding to the spin configuration of energy H. If we consider a decomposition of L into loops l, we obtain

$$H = |L| = \sum_l |l| \tag{5}$$

where $|l|$ is the length of the loop l. The probability p_i^- is

$$p_i^- = \frac{\sum_{L^-\in\bar{E}_L^-} \mathrm{e}^{-\beta|L^-|}}{\sum_{L\in\bar{E}_L} \mathrm{e}^{-\beta|L|}}. \tag{6}$$

In the denominator we sum over all possible sets $L \in \bar{E}_L$ and in the numerator over all subsets $L^- \in \bar{E}_L$ such that an odd number of loops encloses the site i. $\bar{E}_L^-$ denotes the set of all $L^- \in \bar{E}_L$, such that an odd number of loops encloses i, while $\bar{E}_L^+$ denotes the set of all $L^+ \in \bar{E}_L$, such that an even number of loops encloses i. Hence, $\bar{E}_L = \bar{E}_L^+ \cup \bar{E}_L^-$.

We will call l_i the shortest SOS loop belonging to L^- and surrounding i (see figure 5). Each set L^- can be considered as the union of l_i and a suitable set of $\bar{E}_L^+$. Therefore, equation (6) can be written as

$$p_i^- = \frac{\sum_{l_i} \mathrm{e}^{-\beta|l_i|} \sum'_{L^+\in\bar{E}_L^+} \mathrm{e}^{-\beta|L^+|}}{\sum_{L^+\in\bar{E}_L^+} \mathrm{e}^{-\beta|L^+|} + \sum_{L^-\in\bar{E}_L^-} \mathrm{e}^{-\beta|L^-|}}. \tag{7}$$

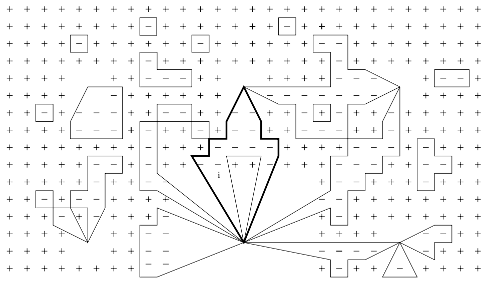

Figure 5. The heavy line denotes the shortest self-avoiding on the sites loop in L surrounding the site i.

$\sum_{l_i}$ denotes the sum over all possible SOS loops l_i enclosing the site i. $\sum'_{L^+\in\bar{E}_L^+}$ is the sum over all $L^+ \in \bar{E}_L^+$ such that $L^+ \cap l_i = \emptyset$ and that, in $l_i \cup L^+$, l_i is the shortest SOS loop surrounding i. Dropping the restriction on the sum $\sum'_{L^+\in\bar{E}_L^+}$, we obtain an overestimate of p_i^-:

$$p_i^- \leqslant \frac{\sum_{l_i} \mathrm{e}^{-\beta|l_i|} \sum_{L^+\in\bar{E}_L^+} \mathrm{e}^{-\beta|L^+|}}{\sum_{L^+\in\bar{E}_L^+} \mathrm{e}^{-\beta|L^+|} + \sum_{L^-\in\bar{E}_L^-} \mathrm{e}^{-\beta|L^-|}} \leqslant \sum_{l_i} \mathrm{e}^{-\beta|l_i|} \leqslant \sum_{k=0}^{\infty} n_i(k)\,\mathrm{e}^{-\beta k}. \quad (8)$$

$n_i(k)$ represents the number of SOS loops of length k enclosing i. Eventually, we can write

$$p_i^- \leqslant \sum_{\alpha} \sum_{k=0}^{\infty} n_i^{\alpha}(k)\,\mathrm{e}^{-\beta k} \quad (9)$$

where $n_i^{\alpha}(k)$ is the number of SOS loops of length k, enclosing i and passing the site α of the dual graph.

Now we take the thermodynamic limit. The theorem will be proved by showing on $n_i^{\alpha}(k)$ an exponential bound independent of i. Up to now we followed step by step the Peierls argument for the Ising model on two-dimensional lattices [14]. However, here the dual graph is more intricate than in the two-dimensional case, since it has unbounded coordination. Hence, the proof of the bound on $n_i^{\alpha}(k)$ requires more work.

4. Tree representation of the loops

In order to count the SOS loops of length k, surrounding i and starting from α, we represent them into a tree. By induction on N we prove that any set of N SOS loops of length k, starting from α', can be represented in a tree satisfying the following properties (TP).

Each site of the tree a is associated with a site of the dual graph $\alpha(a)$. Nearest neighbour sites in the tree correspond to nearest neighbour sites in the dual graph. The correspondence is not one to one; however, if b and c are nearest neighbours of the same site of the tree, then they represent different sites of the dual, i.e. $\alpha(b) \neq \alpha(c)$. Consequently, calling $\tilde{z}_a$ the coordination number of a site of the tree and z_α the coordination of a site of the dual, we have $\tilde{z}_\alpha \leqslant z_{\alpha(a)}$. One site of the tree is called the origin o. We have $\alpha(o) = \alpha'$ and the maximum distance of any sites of the tree from o is k (k will be called the length of the tree). There are exactly N sites at a distance of k steps from o and with each of them we associate, in a one-to-one correspondence, one of the loops. These sites will be called ending points e. $\alpha(e) = \alpha'$ and they are the only ones in the tree with coordination 1. Therefore, the number of loops represented in the tree can be evaluated by counting the ending points. Finally, the sites crossed in the tree by the walk of k steps, connecting the origin to an ending point e, correspond, in a one-to-one relation, to the sites of the dual graph belonging to the loop associated with e.

The set made of any single loop can be represented by a linear chain of $k + 1$ sites and k links. The first site of the chain is the origin and it corresponds to α', the second corresponds to the second site of the loop and so on, and the last site corresponds again to α' and it is the ending point of the tree.

Let us show that, if there is a tree representation for any set of $N - 1$ loops starting from α', we can add to any of these sets a new loop and produce a tree representing this new set. We will proceed by adding a new chain to the tree, relevant to the $N - 1$ loops. The sites of this tree correspond, in both representations, to the same sites of the dual graph and they represent the first $N - 1$ loops. The starting point α' of the new loop is naturally associated with the origin o of the tree. Then, if one of the nearest neighbours of o corresponds in the dual to the second site of the loop, we move to this site of the tree and check if one of its neighbours corresponds to the third site of the loop and so on. Until we find that, at the step q, no neighbour of the site represents the site reached in the $(q + 1)$th step of the loop. Since we are considering SOS loops (we cannot move backward), this site is at distance q from the origin of the tree. Now we add to this site a bifurcation and a new chain of length $k - q$. The first site of this new chain corresponds to the site reached in the $(q + 1)$th step of the loop, the second to the site reached in the $(q + 2)$th step, and the last corresponds again to α'. So, when we add a loop to the set, we add to the tree one ending point naturally associated with the new loop. The new neighbour sites added to the tree are neighbour also in the dual. Furthermore, we never add to any site of the tree a a new neighbour b, such that $\alpha(b)$ is already represented in the tree by a neighbour of a. This way, we produce a tree satisfying TP properties and representing a set of N loops passing α'.

In particular, it is possible to construct the tree representing the set of all SOS loops of length k surrounding i and starting from α. The number of these loops is equal to the number of ending points in the tree. In figure 6 a tree corresponding to all the loops of length 7 passing $\alpha = 0$ and enclosing i is illustrated.

Let us consider a tree of length k and satisfying TP. We call $\bar{e}$ the ending point maximizing the product $\bar{P}_k$ of the coordination numbers of the sites crossed by the k steps walk connecting the origin to $\bar{e}$. We prove by induction on k that the number of ending points T_k is smaller than $\bar{P}_k$.

For $k = 1$, the tree is given by the origin and its $\tilde{z}_o$ neighbours which are the ending points. Hence, both $\bar{P}_1$ and T_1 are equal to $\tilde{z}_o$. Let us now suppose that $T_{k-1} \leqslant \bar{P}_{k-1}$ for any tree of length $k - 1$. We need to show that, given a tree of length k, $T_k \leqslant \bar{P}_k$. Cutting the origin o from the tree of length k, the graph is disconnected into $\tilde{z}_o$ subgraphs. Each of these subgraphs is a tree of length $k - 1$, satisfying TP, whose origin is one of the neighbours of o. We will denote these trees with the index h and call T^h_{k-1} the number of ending points of the tree h. T_k is equal to the sum over h of T^h_{k-1}. Hence,

$$T_k = \sum_{h=1}^{\tilde{z}_o} T^h_{k-1} \leqslant \sum_{h=1}^{\tilde{z}_o} \bar{P}^h_{k-1} \leqslant \tilde{z}_o \max_h \left(\bar{P}^h_{k-1}\right). \tag{10}$$

In the first inequality we use the fact that T^h_{k-1} and $\bar{P}^h_{k-1}$ are, respectively, the number of ending points and the maximum of the products of the coordination numbers in a tree of length $k - 1$. Since the origin belongs to any of the walks connecting o to an ending point, $\bar{P}_k$ can be expressed as the product of z_o and the maximum value of $\bar{P}^h_{k-1}$. Therefore, from (10) one immediately obtains the proof.

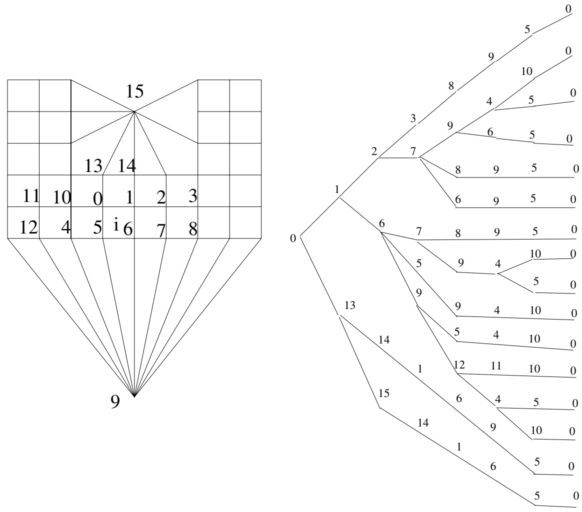

Figure 6. The tree corresponding to all loops of length 7 passing $\alpha = 0$ and enclosing i (the loops are represented in the tree in the clockwise direction).

In the tree representing all the SOS loops of length $|l|$ enclosing i and passing α, we will call $\bar{w}_i^\alpha$ the walk connecting the origin to $\bar{e}$ and call $\bar{l}_i^\alpha$ the loop associated with $\bar{e}$. We have

$$n_i^\alpha(|l|) \leqslant \prod_{a \in \bar{w}_i^\alpha} \tilde{z}_a = \tilde{Z}_{\bar{l}_i^\alpha} \prod_{a \in \bar{w}'^\alpha_i} \tilde{z}_a \leqslant \tilde{Z}_{\bar{l}_i^\alpha} \exp\left(\sum_{\gamma \in \bar{l}'^\alpha_i} \log(z_\gamma) \right). \tag{11}$$

$\tilde{Z}_{\bar{l}_i^\alpha}$ is the coordination number of the tree site m corresponding to the site μ of the loop $\bar{l}_i^\alpha$ with the highest coordination in the dual graph (if there are more than one sites of highest coordination, we arbitrarily choose one of them). $\bar{w}'^\alpha_i$ and $\bar{l}'^\alpha_i$ are obtained by subtracting from the walk $\bar{w}_i^\alpha$ and the loop $\bar{l}_i^\alpha$, respectively, m and μ. In the last inequality, we use the one-to-one correspondence between $\bar{w}'^\alpha_i$ and $\bar{l}'^\alpha_i$ and the fact that the coordination number $\tilde{z}_a$ of the sites of the tree is smaller than or equal to the coordination number $z_{\gamma(a)}$ of the relevant sites of the dual graph.

5. Sparse high connectivity

In the dual graph, the possible values of the coordination numbers are 4×3^m with $m = 0, 1, \ldots$. Hence, we have

$$n_i^\alpha(|l|) \leqslant \tilde{Z}_{\bar{l}_i^p} \exp\left(\sum_m N_{\bar{l}'^\alpha_i}(m)(\log(4) + m\log(3)) \right). \tag{12}$$

$N_{\bar{l}'^\alpha_i}(m)$ is the number of sites in $\bar{l}'^\alpha_i$ with coordination number in the dual graph equal to 4×3^m.

The minimum chemical distance $d(m)$ between two sites of the dual graph with coordination greater than or equal to 4×3^m is

$$d(m) = \begin{cases} 1 & \text{if} \quad m = 0 \\ 2^{m+1} & \text{if} \quad m > 0. \end{cases} \tag{13}$$

Equation (13) represents the fundamental topological property of the dual graph of the Sierpinski carpet which will allow us to prove the exponential bound on $n_i(k)$, even in the absence of bounded coordination. We will call this property sparse high connectivity, since it implies that in the topology of the dual graph generated by the chemical distance, sites with high coordination are far apart.

For a SOS loop l, the sites with coordination $\geqslant 4 \times 3^m$ are less than $1 + |l|/d(m)$. Since in $\bar{l}'^{\alpha}_i$ we have excluded the point of highest coordination, we can overestimate the number of sites of coordination 4×3^m as

$$N_{\bar{l}'^{\alpha}_i}(m) \leqslant \frac{|l|}{d(m)}. \tag{14}$$

From (14) we obtain that, if $m > \log_2(l) - 1 = m_M$, then $N_{\bar{l}'^{\alpha}_i}(m) = 0$. Therefore, m_M represents the maximum value of m on the sites of $\bar{l}'^{\alpha}_i$.

Thus, we have

$$\begin{aligned} n_i^{\alpha}(l) &\leqslant \tilde{Z}_{\bar{l}^{\alpha}_i} \exp\left(|l|\log(4) + l\sum_{m=1}^{m_M} 2^{-m-1}(\log(4) + m\log(3))\right) \\ &\leqslant \tilde{Z}_{\bar{l}^{\alpha}_i} \exp\left(|l|\log(4) + l\sum_{m=1}^{\infty} 2^{-m-1}(\log(4) + m\log(3))\right) \\ &\leqslant \tilde{Z}_{\bar{l}^{\alpha}_i} \exp(K|l|) \end{aligned} \tag{15}$$

where K is a suitable positive, real constant.

6. The site of highest coordination

In this section we will show that $\tilde{Z}_{\bar{l}^{\alpha}_i} \leqslant 10 l^{\log_2(3)}$. If $z_\mu \leqslant 10 l^{\log_2(3)}$ (μ is the site in the dual graph with highest coordination), the inequality is trivially satisfied. Otherwise, we will use the fact that $\tilde{Z}_{\bar{l}^{\alpha}_i}$ is the coordination of a site of a tree representing SOS loops surrounding i. Two sites of the tree are neighbours only if the corresponding sites of the dual are neighbours and there exists a SOS loop passing through them and enclosing i. Therefore, the bound on $\tilde{Z}_{\bar{l}^{\alpha}_i}$ is proved by showing that the $Z_{\mu,\bar{l}_i} \leqslant 10 l^{\log_2(3)}$, where $Z_{\mu,\bar{l}_i}$ represents the number of nearest neighbours of μ, which can be crossed by a SOS walk of length $|l|$, passing μ and enclosing i.

Let ν and ρ be the nearest neighbours of μ. We define the distance along the boundary $\bar{r}_{\nu,\rho}$ as in figure 7. In figure 7, the plaquette of the site μ is also defined.

Let us consider a SOS loop $l^{\nu,\rho}$ of length $|l|$ passing μ and its nearest neighbours ρ and ν. We will prove that

$$\bar{r}_{\nu,\rho} \leqslant 2|l|^{\log_2(3)}. \tag{16}$$

Furthermore, if $l^{\nu,\rho}$ encloses the site of the carpet i, from the planarity of the dual graph we have the following property: no loop of length $|l|$ passing μ and enclosing i can pass through any site τ of the plaquette of μ, such that τ is located with respect to ρ in the opposite direction of ν at a distance $\bar{r}_{\tau,\rho} > 4 \times l^{\log_2(3)}$.

An analogous property holds for the sites of the plaquette at a distance greater than $4 \times l^{\log_2(3)}$ from ν in the direction opposite to ρ. Therefore, since $\bar{r}_{\nu,\rho} \leqslant 2|l|^{\log_2(3)}$, the number $Z_{\mu,\bar{l}_i}$ of sites belonging to the plaquette of μ and crossed by a loop passing μ and surrounding i is smaller than $10|l|^{\log_2(3)}$. This way we obtain the bound on $\tilde{Z}_{\bar{l}^{\alpha}_i}$.

Let us prove (16). The dual graph is naturally embedded in a two-dimensional Euclidean space. Since the plaquettes are convex subsets of this space, the length of the border $\bar{r}_{\nu,\rho}$ turns

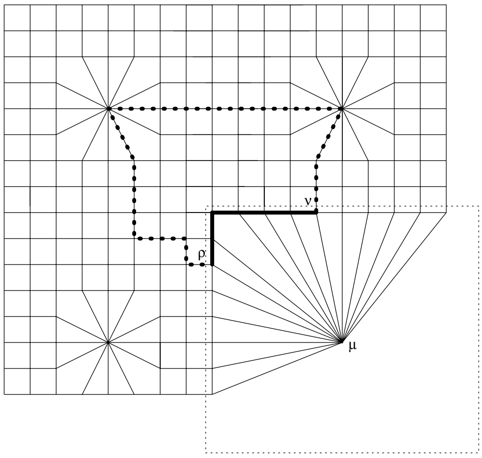

Figure 7. The plaquette of the site μ is given by the region denoted by the dashed line. The sites within this region (excluded μ) are the sites belonging to the plaquette. The distance along the boundary between the sites of the plaquette ν and ρ is the length of the line plotted with the heavy line. In this example $\bar{r}_{\rho,\nu} = 6$. By the dotted lines we denoted a walk in the dual graph between ν and ρ of length 18.

out to be smaller than the length, in the Euclidean space, of any walk from ν to ρ which does not pass through the plaquette itself. In particular, it is smaller than the length of $l'^{\nu,\rho}$. $l'^{\nu,\rho}$ is the walk obtained by subtracting μ from the loop $l^{\nu,\rho}$. Now, a step in the dual graph, crossing a site with coordination z_ρ, covers, in the Euclidean space, a distance smaller than $z_\rho/2$. We have

$$\bar{r}_{\nu,\rho} \leqslant \frac{1}{2}\sum_{\rho\in l'^{\nu,\rho}} z_\rho \leqslant 2\sum_{m}^{m_M} N_{l'^{\nu,\rho}}(m)3^m \leqslant 2l + l\sum_{m}^{\log_2(|l|)-1}\left(\frac{3}{2}\right)^m \leqslant 2l^{\log_2(3)}. \tag{17}$$

$N_{l'^{\nu,\rho}}(m)$ is the number of sites in $l'^{\nu,\rho}$ with coordination 4×3^m. In equation (17), $N_{l^{\nu,\rho}}(m)$ has been estimated using inequality (14).

Now we need to prove that no loop of length $|l|$ passing τ and μ can enclose the site i. We suppose, *ad absurdum*, the existence of this loop and we call it $l^{\tau,\sigma}$ (σ is the other site of the loop belonging to the plaquette of μ). Since τ and σ are connected by a loop of length $|l|$, from (16) we have $\bar{r}_{\tau,\sigma} < 2|l|^{\log_2(3)}$. Hence, $\bar{r}_{\rho,\sigma} > 2|l|^{\log_2(3)}$. If $l^{\tau,\sigma}$ also surrounded i, for the planarity of the dual graph, there should exist two sites γ and δ belonging both to $l^{\tau,\sigma}$ and to $l^{\nu,\rho}$ (see figure 8 for an illustration).

We denote by $|\gamma,\delta|_{l^{\tau,\sigma}}$ the length of the part of the walk $l'^{\tau,\sigma}$ within the sites γ and δ. We have

$$\begin{aligned} &|\rho,\delta|_{l^{\nu,\rho}} + |\delta,\gamma|_{l^{\nu,\rho}} + |\gamma,\nu|_{l^{\nu,\rho}} = |l| - 2 \\ &|\sigma,\delta|_{l^{\tau,\sigma}} + |\delta,\gamma|_{l^{\tau,\sigma}} + |\gamma,\tau|_{l^{\tau,\sigma}} = |l| - 2 \end{aligned} \tag{18}$$

and then

$$|\rho,\delta|_{l^{\nu,\rho}} + |\gamma,\nu|_{l^{\nu,\rho}} + |\sigma,\delta|_{l^{\tau,\sigma}} + |\gamma,\tau|_{l^{\tau,\sigma}} \leqslant 2(|l|-2). \tag{19}$$

On the other hand, since $\bar{r}_{\nu,\tau} > |l|^{\log_2(3)}$ and $\bar{r}_{\rho,\sigma} > |l|^{\log_2(3)}$, from (16) we get

$$\begin{aligned} &|\rho,\delta|_{l^{\nu,\rho}} + |\sigma,\delta|_{l^{\tau,\sigma}} > |l| - 2 \\ &|\nu,\gamma|_{l^{\nu,\rho}} + |\tau,\gamma|_{l^{\tau,\sigma}} > |l| - 2 \end{aligned} \tag{20}$$

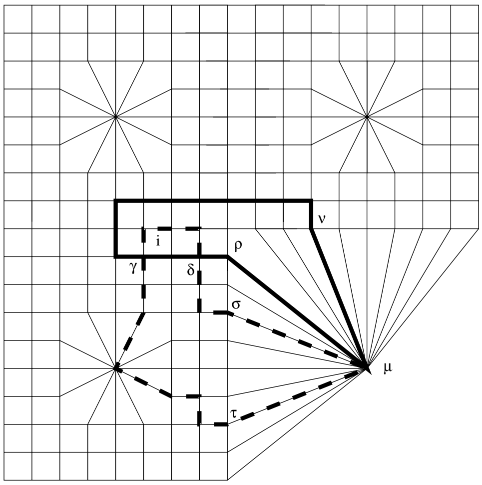

Figure 8. By the heavy continuous line we denote $l^{\nu,\rho}$, and by the dashed line $l^{\tau,\sigma}$.

and then

$$|\rho, \delta|_{l^{\nu,\rho}} + |\sigma, \delta|_{l^{\tau,\sigma}} + |\nu, \gamma|_{l^{\nu,\rho}} + |\tau, \gamma|_{l^{\tau,\sigma}} > 2(|l| - 2). \tag{21}$$

Equation (21) cannot hold together with equation (19). Therefore, assuming the existence of the site τ, we obtain the contradiction we wanted to prove. Hence, we have concluded the proof of the bound on $\tilde{Z}_{\bar{l}_i^\alpha}$.

7. Euclidean length of a loop

With the bound on $\tilde{Z}_{\bar{l}_i^\alpha}$ proved in the previous section, we obtain that the probability for the site i to have negative magnetization can be estimated as

$$p_i^- \leqslant \sum_{|l|=0}^{\infty} N_i(|l|) 10 |l|^{\log_2(3)} \, \mathrm{e}^{(K-\beta)|l|}. \tag{22}$$

$N_i(|l|)$ is the number of points in the dual lattice which can be crossed by a SOS of length $|l|$ which encloses i.

An estimate of $N_i(|l|)$ can be obtained by using again the fact that the dual graph is embeddable in a two-dimensional Euclidean space. In particular, in a walk in the dual graph, a step crossing the site α covers in the Euclidean space a distance smaller than or equal to $z'_\alpha = \bar{r}_{\alpha^-,\alpha^+}$, where α^- and α^+ are, respectively, the sites preceding and following α within the walk. Let us consider l_i SOS loop in the dual graph of length $|l|$ and enclosing the site i of the carpet. We call $d(l_i)$ the distance covered by this loop in the Euclidean space, and we get

$$d(l_i) \leqslant \sum_{\alpha \in l_i} z'_\alpha \leqslant Z'_{l_i} + \sum_{\alpha \in l'_i} z_\alpha \tag{23}$$

In the sum we separate the contribution of the site with highest coordination number and use the simple property that $z'_\alpha < z_\alpha$. $Z'_{\bar{l}_i}$ represents the distance along the boundary between the sites preceding and following the site of highest coordination number in a walk of length $|l|$. In the previous section, we proved that $Z'_{\bar{l}_i} \leqslant 2l^{\log_2(3)}$. Furthermore, the sum over all sites belonging to l'_i has been already evaluated in (17). Therefore, we have that

$$d(l_i) \leqslant 8|l|^{\log_2(3)}. \tag{24}$$

The number of sites of the dual graph, distance from i in the Euclidean space less than $d(l_i)$, is smaller than $\pi d^2(l_i)$. Therefore we obtain that $N_i(l) \leqslant \pi(8|l|^{\log_2(3)})^2$. Hence

$$p_i^- \leqslant \sum_{|l|=0}^{\infty} C|l|^{3\log_2(3)}\, \mathrm{e}^{(K-\beta)|l|} \tag{25}$$

where C is a suitable real constant. For small enough values β, the sum in (25) is smaller than $\epsilon < 1/2$. Thus, we obtain the uniform bound on p_i^- proving the existence of spontaneous magnetization at finite temperature.

Acknowledgments

The author is grateful to R Burioni and D Cassi for useful comments and discussions.

References

[1] Mandelbrot B B 1975 *The Fractal Geometry of Nature* (San Francisco, CA: Freeman)
[2] Gefen Y, Aharony A and Mandelbrot B B 1984 *J. Phys. A: Math. Gen.* **17** 1277
Wu Y K and Hu B 1987 *Phys. Rev.* A **35** 1404
Bonnier B, Leroyer Y and Meyers C 1988 *Phys. Rev.* B **37** 5205
[3] Hattori K, Hattori T and Watanabe H 1985 *Phys. Rev.* A **32** 3730
Lai P and Goldschmit 1987 *J. Phys. A: Math. Gen.* **20** 2159
Chame A and Costa U M S 1990 *J. Phys. A: Math. Gen.* **23** L1127
Kim M H, Yoon D H and Kim I 1993 *J. Phys. A: Math. Gen.* **26** 5655
[4] Bonnier B, Leroyer Y and Meyers C 1989 *Phys. Rev.* B **40** 8961
Fabio D A, Arão Reis A and Riera R 1994 *Phys. Rev.* E **49** 2579
[5] Banhot G, Neuberger H and Shapiro J A 1984 *Phys. Rev. Lett.* **58** 6386
Angles D'Auriac J C and Rammal R 1986 *J. Phys. A: Math. Gen.* **19** L655
Monceanu P, Perreau M and Hebért F 1998 *Phys. Rev.* B **58** 6386
Zheng G P and Li M 2000 *Phys. Rev. E* **62** 6253
Pruessner G, Loison D and Schotte K D 2001 *Phys. Rev.* B **64** 134414
[6] Dasgupta R, Ballabh T K and Tarafdar S 1999 *J. Phys. A: Math. Gen.* **32** 6503
[7] Gefen Y, Aharony A and Mandelbrot B B 1984 *J. Phys. A: Math. Gen.* **17** 435
Higuchi Y and Yoshida N 1996 *J. Stat. Phys.* **84** 295
[8] Rammal R 1984 *J. Phys. (Paris)* **45** 191
Stinchcombe R B 1990 *Phys. Rev.* B **41** 2510
[9] Barlow M T and Bass R F 1999 *Random Walks and Discrete Potential Theory* ed M Piccardello and W Woess (Cambridge: Cambridge University Press) p 26
[10] Alexander S and Orbach R 1987 *J. Phys. (Paris) Lett.* **92** 108
Hattori K, Hattori T and Watanabe H 1987 *Prog. Theor. Phys. Suppl.* **92** 108
Burioni R and Cassi D 1997 *Mod. Phys. Lett.* B **11S** 1095
[11] Cassi D 1996 *Phys. Rev. Lett.* **76** 2941
[12] Shinoda M 2002 *J. Appl. Probab.* **39** 1
[13] Peierls R 1936 *Proc. Camb. Phil. Soc.* **32** 477
[14] Griffiths R B 1964 *Phys. Rev.* **136** 437
Huang K 1987 *Statistical Mechanics* (New York: Wiley)
[15] Chayes J T, Chayes L and Fröhlich J 1985 *Commun. Math. Phys.* **100** 399